2020

MATHEMATICS — HONOURS

Paper : CC-7

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real number

Group - A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify ; (1+1)×10

(a) If x, x^2, x^3 are three linearly independent solutions of a third-order differential equation, then the Wronskian W of the functions has value

(i) $W = 2x^3$ (ii) $W = x^3$ (iii) $W = x^2$ (iv) $W = 2x^2$.

(b) One of the points which lies on the solution curve of the differential equation $(y - x)dx + (x + y)dy = 0$ with given condition $y(0) = 1$ is

(i) $(1, -2)$ (ii) $(2, -1)$ (iii) $(2, 1)$ (iv) $(-1, 2)$.

(c) If the integrating factor of $(x^7y^2 + 3y)dx + (3x^8y - x)dy = 0$ is x^my^n, then

(i) $m = -7$, $n = 1$ (ii) $m = 1$, $n = -7$ (iii) $m = 0$, $n = 0$ (iv) $m = 1$, $n = 1$.

(d) Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0, y(0) = 1, \left.\frac{dy}{dx}\right|_{x=0} = -1$, then $y(x)$ attains its maximum value at

(i) $ln\frac{4}{3}$ (ii) $ln\frac{3}{4}$ (iii) $ln\frac{1}{2}$ (iv) none of these.

(e) Consider the differential equation $a\frac{dy}{dx} + by = ce^{-\lambda x}$, where a, b, c are positive constants and λ is a non-negative constant. Then every solution of the differential equation approaches to $\frac{c}{b}$ as $x \to +\infty$ when

(i) $\lambda > 0$ (ii) $\lambda = 0$ (iii) $\lambda = \frac{b}{a}$ (iv) $\lambda = \frac{a}{b}$.

(f) Which one of the following is correct for the linear differential equation $\left(x^2+x\right)\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}+2y=0$?

(i) 0 is an ordinary point (ii) -1 is a regular singular point

(iii) -1 is an irregular singular point (iv) 0 is an irregular singular point.

(g) The initial value problem $x\frac{dy}{dx}=y,\ y(0)=0,\ x\geq 0$ has

(i) no solution (ii) a unique solution

(iii) exactly two solutions (iv) uncountably many solutions.

(h) The double limit $\lim\limits_{(x,y)\to(0,0)}\frac{|x|}{y^2}e^{-\frac{|x|}{y^2}}$

(i) does not exist (ii) exist and equal to 0

(iii) exist and equal to 1 (iv) exist and equal to -1.

(i) Consider the function $f(x, y) = x^2 - 4xy + 4y^2 + 2x^4 + 3y^4$, then

(i) f has no extrema at (0, 0)

(ii) f has maximum value at (0, 0) which is 0

(iii) f has maximum value at (0, 0) which is 1

(iv) f has minimum value at (0, 0) which is 0 .

(j) Let $T(x, y, z) = xy^2 + 2z - x^2z^2$ be the temperature at the point (x, y, z). The unit vector in the direction in which the temperature decreases most rapidly at $(1, 0, -1)$ is

(i) $-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (ii) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

(iii) $\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}$ (iv) $-\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$.

Group - B

(Marks : 30)

Answer ***any six*** questions. 5×6

2. Show that a constant K can be found so that $(x + y)^K$ is an integrating factor of

$$(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0$$

and hence solve the equation.

3. Reduce the equation $x^3p^2 + x^2yp + a^3 = 0$ to Clairaut's form by the substitution $y = u$ and $x = \frac{1}{v}$ and obtain the complete primitive.

4. Solve using the method of undetermined coefficients, the equation with initial conditions,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x, \quad y(0) = 2 \text{ and } y'(0) = 4.$$

5. Solve by the method of variation of parameters the equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x, (x > 0).$

6. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$ by changing the independent variables.

7. Use D-operator to solve :

$$\frac{d^2y}{dx^2} - y = x\sin x + \left(1 + x^2\right)e^x$$

8. Show that the equation of the curve, whose slope at any point (x, y) is equal to $xy(x^2y^2 - 1)$ and which passes through the point (0, 1) is $x^2y^2 = 1 - y^2$.

9. Solve for x from the system of equations

$$\frac{dx}{dt} + 4x + 3y = t$$
$$\frac{dy}{dt} + 2x + 5y = e^t$$

10. Consider the plane autonomous system

$$\frac{dx}{dt} = 2x + y, \quad \frac{dy}{dt} = 3x + 4y$$

Find the general solution of the system. State the nature of the critical point of the system. Discuss its stability. Draw a phase portrait of the system.

11. Solve the equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 + 2\right)y = 0$ in series about the ordinary point $x = 0$.

Group - C

(Marks : 15)

Answer ***any three*** questions.

12. Define limit point of a subset of $\mathbb{R}\times\mathbb{R}$. If $B = \{(a, 0); a \in \mathbb{R}\}$. Show that B is a closed set but not open in $\mathbb{R}\times\mathbb{R}$. 5

13. State the sufficient conditions for differentiability of a function $f: \mathbb{R}^2 \to \mathbb{R}$. Examine whether the sufficient conditions of differentiability are satisfied for the following function $f(x, y)$ and hence comment on differentiability of $f(x, y)$ at (0, 0) where $f(x, y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & , \text{ when } x^2+y^2 \neq 0 \\ 0 & , \text{ when } x^2+y^2 = 0. \end{cases}$ 1+4

14. If z is a function of two variables x and y and $x = c \cosh u \cos v$, $y = c \sinh u \sin v$ (c is a real number), show that

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{c^2}{2}(\cosh 2u - \cos 2v)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$ 5

15. Find all critical points of the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 - 3x - 12y + 40$ for $(x, y) \in \mathbb{R}^2$. Also examine whether the function f attains a local maximum or a local minimum at each of these critical points. 5

16. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, by the method of Lagrange's multipliers. 5
